Int. J. Heat Mass Transfer. Vol. 35, No. 3, pp. 635–639, 1992
Printed in Great Britain

0017-9310/92 $5.00+0.00
Pergamon Press plc

Numerical investigation of turbulent plane and buoyant jets†

O. G. MARTYNENKO and V. N. KOROVKIN
A. V. Luikov Heat and Mass Transfer Institute, Minsk, 220072, Byelorussia

Abstract—The k–ε model of turbulence is used for calculating dynamical and thermal fields in plane turbulent vertical jets in a uniform stagnant environment. A new effective approach to the solution of the governing system of partial differential equations (continuity, conservation of momentum, heat, turbulent kinetic energy and its dissipation rate) is suggested which is based on the introduction of mathematical variables. Comparison is made between the results of the present calculations and the experimental data of other authors, and satisfactory agreement is found.

INTRODUCTION

TRADITIONALLY, free convection is one of the most actual fields of modern thermophysics. This is ascribed to the important role of Archimedes forces in the occurrence of different hydrodynamic and thermal processes in nature and technology. The jet liquid and gas flows are no exceptions in this respect. Actually, investigations of turbulent jets in the presence of buoyancy effects, which started from the late 1940s [1], have turned at present into one of the most rapidly developing problems of hydromechanics, attracting the attention of specialists in diverse areas of research such as meteorology, oceanology, power engineering, etc.

However, in contrast to forced jet flows, the problems of studying buoyant jets are substantially more complicated both from the viewpoint of the procedure and technique of the experiment and the possibility of a theoretical solution of the problem. When, for turbulent flows without account of buoyancy forces, one succeeds in applying the similarity theory to correlate the results of laboratory measurements with the use of a small number of empirical constants, then in the case of jet development in the field of Archimedes forces the number of dimensionless groups increases and, as a rule, a complete similarity is not attained. Though, of course, in specific cases one manages, with an accuracy sufficient for engineering calculations, to use dimensionless relations for analyzing the averaged characteristics of the flow or carrying out an approximate simulation [2, 3]. This is responsible for the attention having been turned to a numerical analysis within the frameworks of diverse models of turbulence [4, 5]. Since the greater portion of works dealing with the study and prediction of the processes of turbulent transport of momentum and heat in jets is based on the results of numerical experiments, the development of effective and reliable numerical schemes for solving the problems posed becomes a particularly urgent problem, very important for engineering practice.

In the present paper a new method is suggested for calculating a turbulent plane vertical jet within the framework of the standard buoyancy-extended k–ε turbulence model. The method consists essentially of the use of mathematical variables that simplify the governing system of equations and make it convenient for numerical integration. Comparison of the results of calculations with the experimental data of other authors has been made.

GOVERNING EQUATIONS

Consider a vertical turbulent jet issuing from a plane slit of width r_0 with a velocity u_0, initial temperature T_0 and density ρ_0 into a non-stratified stationary space ($\rho_0 < \rho_\infty$). The coordinate origin is located in the middle of the nozzle. The x axis is directed along the jet and the y axis runs normal to it. Let u and v denote the averaged velocities. Then, the starting equations for the developed motion and heat transfer (buoyancy) within the framework of the turbulent boundary layer model, with account taken of the Boussinesq approximation, can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(-\langle u'v'\rangle)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\partial}{\partial y}(-\langle v'T'\rangle). \quad (1)$$

As is usually adopted for thin shear layers, the term with normal stresses is discarded from the momentum

† Dedicated to Professor Dr.-Ing. Dr.-Ing.e.h, Ulrich Grigull.

NOMENCLATURE

b	jet half-width determined by coordinate y at which $u = u_m/2$ (or $\Delta T = \Delta T_m/2$)
F	Froude number, $u_0^2/g\beta\Delta T_0 r_0$
g	gravity acceleration
k	kinetic energy of turbulence
r_0	plane jet width
T	averaged temperature, $\Delta T = T - T_\infty$
T'	temperature fluctuation
u, v	averaged velocity components in directions x and y, respectively
u', v'	components of fluctuating velocity
x, y	directions along jet axis.

Greek symbols

β	coefficient of thermal expansion
ε	dissipation of kinetic energy of turbulence
ρ	density
σ_t	turbulent Prandtl number.

Subscripts

∞	surrounding medium
0	at jet exit
c	on jet axis
m	maximum value
1	dimensionless value.

equation, while the energy equation does not involve the streamwise heat flux component.

Due to the uncertainty of the quantities $\langle u'v' \rangle$ and $\langle v'T' \rangle$, standing respectively for the Reynolds stress and turbulent heat flux, the system of equations (1) is not closed and its solution is naturally possible only with the use of any turbulence model. In the present work, the local values of the kinetic energy of turbulence k and of the rate of its dissipation ε, are taken as the basic quantities determining the turbulent transport. These values satisfy the following model equations:

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \frac{1}{\sigma_k}\frac{\partial}{\partial y}\left(v_t\frac{\partial k}{\partial y}\right) + v_t\left(\frac{\partial u}{\partial y}\right)^2 - \varepsilon$$

$$u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} = \frac{1}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left(v_t\frac{\partial \varepsilon}{\partial y}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}v_t\left(\frac{\partial u}{\partial y}\right)^2 - c_{\varepsilon 2}\frac{\varepsilon^2}{k}. \quad (2)$$

The algebraic expressions for the shear and transverse heat flux are connected to k and ε by the so-called Kolmogorov–Prandtl relation:

$$\langle u'v' \rangle = -v_t\frac{\partial u}{\partial y}, \quad \langle v'T' \rangle = -\frac{v_t}{\sigma_t}\frac{\partial T}{\partial y} \quad v_t = c_\mu\frac{k^2}{\varepsilon}. \quad (3)$$

Equations (1)–(3) form a closed system, which corresponds to the k–ε model of turbulence and involves five coefficients

$$c_\mu(\infty) = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3,$$
$$c_{\varepsilon 1} = 1.44, \quad c_{\varepsilon 2} = 1.92. \quad (4)$$

However, in contrast to the standard value, the value of the quantity c_μ is not constant but is an empirical function of the Froude number [6]

$$c_\mu = 0.09\left[1 + \frac{4}{9}\left(1 + \tanh\left(2\log\frac{1}{F} + 3\right)\right)\right]. \quad (5)$$

This is due to the fact that at large values of the streamwise coordinate x there is a quantitative difference between experimental and numerical results: at $c_\mu = 0.09$, model (1)–(4) correctly describes the experimentally observable relationships, but the predicted values for the jet flow become overestimated by 15–20%. For a free shear layer in the absence of Archimedes forces ($F = \infty$), relation (5) yields $c_\mu(\infty) = 0.09$, and the model is exactly reduced to the standard one.

For a full statement of the problem, equations (1)–(5) should also be augmented with information about the initial and boundary conditions

$$x = 0\begin{cases} u = u_0, \quad T = T_0, \quad \varepsilon = \varepsilon_0, \quad k = k_0 & \text{when } 0 \leqslant y < \frac{r_0}{2} \\ u = 0, \quad T = T_\infty, \quad \varepsilon = 0, \quad k = 0 & \text{when } \frac{r_0}{2} \leqslant y < \infty \end{cases}$$

$$x > 0\begin{cases} v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial \varepsilon}{\partial y} = \frac{\partial k}{\partial y} = 0 & \text{at } y = 0 \\ u \to 0, \quad T \to 0, \quad k \to 0, \quad \varepsilon \to 0 & \text{for } y \to \infty. \end{cases} \quad (6)$$

CALCULATION TECHNIQUE

Usually the equations for averaged flow (1), together with equations (2)–(6), after the transition to corresponding dimensionless variables, are integrated numerically by the finite-difference technique. In this case the flow region is overlaid with a rectangular grid on the plane x, y with the step Δx, Δy; whereas the partial differential equations at all the inner nodes of this grid are replaced by the difference ones. Further, the resulting non-linear system of algebraic equalities is solved by iteration with the aid of the factorization technique. The test numerical calculations show that when iterations are not carried out for solving each of the finite-difference equations, the overall iterative process turns out to be non-convergent.

Below, an alternative approach to the analysis of equations (1)–(6) is suggested. New coordinates are introduced

$$X = X, \quad \eta = 2\int_0^Y U\theta \, \mathrm{d}Y. \tag{7}$$

Then, by virtue of the formulae of transition from the old variables X and Y to the new ones,

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial X} + \left(4\frac{c_\mu}{\sigma_t}\frac{K^2}{E}U\theta\frac{\partial\theta}{\partial\eta} - 2V\theta\right)\frac{\partial}{\partial\eta}$$

$$\frac{\partial}{\partial Y} = 2U\theta\frac{\partial}{\partial\eta},$$

the following equations are obtained:

$$\frac{\partial U}{\partial X} = 4c_\mu\frac{K^2\theta^2}{E}\left[U\frac{\partial^2 U}{\partial\eta^2} + \left(\frac{\partial U}{\partial\eta}\right)^2 + 2\frac{U}{K}\frac{\partial U}{\partial\eta}\frac{\partial K}{\partial\eta} - \frac{U}{E}\frac{\partial U}{\partial\eta}\frac{\partial E}{\partial\eta} + \left(1-\frac{1}{\sigma_t}\right)\frac{U}{\theta}\frac{\partial U}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right] + \frac{1}{F}\frac{\theta}{U}$$

$$\frac{\partial K}{\partial X} = 4\frac{c_\mu}{\sigma_k}\frac{K^2\theta^2}{E}\left[U\frac{\partial^2 K}{\partial\eta^2} + 2\frac{U}{K}\left(\frac{\partial K}{\partial\eta}\right)^2 - \frac{U}{E}\frac{\partial E}{\partial\eta}\frac{\partial K}{\partial\eta} + \left(1-\frac{\sigma_k}{\sigma_t}\right)\frac{U}{\theta}\frac{\partial\theta}{\partial\eta}\frac{\partial K}{\partial\eta} + \frac{\partial U}{\partial\eta}\frac{\partial K}{\partial\eta} + \sigma_k U\left(\frac{\partial U}{\partial\eta}\right)^2\right] - \frac{E}{U}$$

$$\frac{\partial E}{\partial X} = 4\frac{c_\mu}{\sigma_\varepsilon}\frac{K^2\theta^2}{E}\left[U\frac{\partial^2 E}{\partial\eta^2} + \left(1-\frac{\sigma_\varepsilon}{\sigma_t}\right)\frac{U}{\theta}\frac{\partial\theta}{\partial\eta}\frac{\partial E}{\partial\eta} + \frac{\partial U}{\partial\eta}\frac{\partial E}{\partial\eta} + 2\frac{U}{K}\frac{\partial K}{\partial\eta}\frac{\partial E}{\partial\eta} - \frac{U}{E}\left(\frac{\partial E}{\partial\eta}\right)^2 + \sigma_\varepsilon c_{\varepsilon 1}\frac{UE}{K}\left(\frac{\partial U}{\partial\eta}\right)^2\right] - c_{\varepsilon 2}\frac{E^2}{KU}$$

$$\frac{\partial\theta}{\partial X} = 4\frac{c_\mu}{\sigma_t}\frac{K^2\theta^2}{E}\left[U\frac{\partial^2\theta}{\partial\eta^2} + \frac{\partial U}{\partial\eta}\frac{\partial\theta}{\partial\eta} + 2\frac{U}{K}\frac{\partial K}{\partial\eta}\frac{\partial\theta}{\partial\eta} - \frac{U'}{E}\frac{\partial E}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right] \tag{8}$$

where

$$U = \frac{u}{u_0}, \quad K = \frac{k}{u_0^2}, \quad E = \frac{\varepsilon r_0}{u_0^3}, \quad \theta = \frac{T - T_\infty}{T_0 - T_\infty},$$

$$X = \frac{x}{r_0}, \quad Y = \frac{y}{r_0}.$$

The advantage gained by the proposed replacement of variables is that in the case of numerical solution it ensures an automatic fulfilment of the enthalpy flux conservation law, which is defined as

$$Q_0 = \rho C_p \int_{-\infty}^{+\infty} u\Delta T \, \mathrm{d}y$$

and also transforms the infinite region of integration into a finite width band. Then, the flow field, which is now a rectangle (at uniform initial velocity and temperature profiles $0 \leqslant X < \infty$ and $0 \leqslant \eta \leqslant 1$), is divided into n bands, and on each of the lines η_i ($i = 1, 2, \ldots, n+1$) the derivatives with respect to η are replaced by their three-point central-difference analogues so that the original partial differential equations are converted into a system of ordinary differential equations. When the initial conditions are prescribed for velocity, temperature, kinetic energy of turbulence, its dissipation rate at the nozzle cut (uniform fields were adopted in calculations)

$$U_0 = 1, \quad \theta_0 = 1, \quad K_0 = 2\times 10^{-2},$$

$$E_0 = 1.6\times 10^{-3} \ (X = 0)$$

or in a certain section X^* (they can be model conditions or those taken from experiment), the Cauchy problem is obtained which is integrated by the standard Runge–Kutta technique with an automatic selection of the step along the dimensionless coordinate $X = x/r_0$. In this connection, the proposed computational scheme is free from the drawbacks inherent in the numerical analysis of turbulent boundary layer equations by the methods of straight lines. As is known, the version $X =$ const. requires the solution of the boundary-value problem for a system of ordinary differential equations (the shooting method is used), which usually depends strongly on the boundary conditions lacking at $Y = 0$.

Suppose the problem has been solved, i.e. the expressions of $U(X, \eta)$, $\theta(X, \eta)$, $K(X, \eta)$ and $E(X, \eta)$ have been found. Then, to bring the solution to an end, the following equality should be used:

$$Y = \frac{1}{2}\int_0^\eta \frac{\mathrm{d}\eta}{U\theta}$$

which ensures the transition of the unknown functions to the physical plane.

NUMERICAL RESULTS

Numerical integration of the particular problem at different values of Prandtl and Froude numbers allows one to obtain detailed information on the dynamics of jet flow development in the form of the fields of the averaged velocities and temperatures and also of the fluctuational characteristics of turbulence. However, the main aim of the present paper is to check the applicability of the proposed calculation method for predicting the laws governing turbulent momentum and heat transfer in real jets. Therefore, the discussion will be limited to the analysis of the data of calculation corresponding to this aim. Table

Table 1. Predicted and experimental (in parentheses) [7] values of the mean characteristics of a plane buoyant jet ($F = 20$)

x/r_0	u_c/u_0	$\Delta T_c/\Delta T_0$	$b_{0.5u}/r_0$	$b_{0.5\Delta T}/r_0$
20	0.827 (0.813)	0.305 (0.371)	2.43 (2.48)	3.04 (3.02)
30	0.806 (0.797)	0.224 (0.270)	3.42 (3.44)	4.22 (3.92)
40	0.798 (0.758)	0.177 (0.206)	4.37 (4.32)	5.38 (4.92)
50	0.794 (0.726)	0.146 (0.161)	5.31 (6.50)	6.52 (7.54)
60	0.792 (0.778)	0.125 (0.138)	6.24 (7.04)	7.66 (7.60)

1 presents calculated ($\sigma_t = 0.5$) and experimental [7] variation of the velocity u_c/u_0 and temperature $\Delta T_c/\Delta T_0$ along the flow axis as well as the half-width of the dynamical and thermal boundary layers in a plane buoyant jet at $F = 20$. A satisfactory agreement between theory and experiment for all four quantities can be noted on the whole. Usually, to correlate the results of numerical investigation, a special unified form of calculated data representation is used. It is based on the self-similarity properties of the jet flow structure at large values of the streamwise coordinate and the corresponding normalization of main characteristics. It is found that when $x_1 > 5$ the predicted changes in the flow parameters for all the versions ($1 \leqslant F \leqslant 500$) can be approximated by the following power-law relations:

$$u_1 = 2.15,\ \theta_1 = 2.70x_1^{-1},\ b_u = 0.106x,$$
$$b_\theta = 0.130x,\ u_1 = U_c F^{1/3},\ \theta_1 = \theta_c F^{1/3},$$
$$x_1 = XF^{-2/3}. \qquad (9)$$

Comparison of equations (9) with the generalized experimental data from ref. [7]

$$u_1 = 2.13,\ \theta_1 = 2.56x_1^{-1},\ b_u = 0.11x,\ b_\theta = 0.133x,$$

again reveals their satisfactory agreement. As to the turbulence characteristics, the numerical data

$$\frac{\langle u'v'\rangle_m}{u_c^2} = 0.030,\quad \frac{\langle v'T'\rangle_m}{u_c\Delta T_c} = 0.046,\quad \frac{k_c}{u_c^2} = 0.046$$

indicate that the latter attain their asymptotic values at a somewhat greater distance (approximately when $x_1 > 8$) from the jet source than parameters (9). The averaging of the experimental values of $\langle u'\rangle$, $\langle v'\rangle$, shear $\langle u'v'\rangle$ and of the transverse turbulent heat flux $\langle v'T'\rangle$ within the range $40 \leqslant X \leqslant 60$ gave [7]

$$\frac{\langle u'v'\rangle_m}{u_c^2} = 0.031,\quad \frac{\langle v'T'\rangle_m}{u_c\Delta T_c} = 0.045,\quad \frac{k_c}{u_c^2} = 0.0907.$$

Since the value of $\langle w'\rangle$ was not measured in laboratory investigations [7], the kinetic energy of turbulence was calculated from the equation

$$k = \tfrac{1}{2}(\langle u'^2\rangle + \langle v'^2\rangle + \langle w'^2\rangle) = \tfrac{1}{2}(\langle u'^2\rangle + 2\langle v'^2\rangle).$$

It is also of interest to compare the results of numerical integration of equations (8) and of laboratory measurements in the case of the forced jet development ($F = \infty$).

Calculations were started at the uniform fields of u_0, θ_0, K_0 and E_0 at the exit from the slit ($\sigma_t = 0.6$) and were continued until the initial profiles became self-similar. The prediction of the asymptotic behaviour of the average parameters

$$U_c = 2.40X^{-1/2},\quad \theta_c = 2.14X^{-1/2},$$
$$b_u = 0.116x,\quad b_\theta = 0.154x,$$

and also of the turbulent characteristics of the plane jet in the case of

$$\frac{\langle u'v'\rangle_m}{u_c^2} = 0.023,\quad \frac{\langle v'T'\rangle_m}{u_c\Delta T_c} = 0.029,\quad \frac{k_c}{u_c^2} = 0.067,$$

turns out to be in adequate agreement with the experimental one [8, 9]

$$U_c = 2.44X^{-1/2},\ \theta_c = 2.27X^{-1/2},\ b_u = 0.11x,$$
$$b_\theta = 0.167x,\ \frac{\langle u'v'\rangle_m}{u_c^2} = 0.026,\ \frac{\langle v'T'\rangle_m}{u_c\Delta T_c} = 0.018,$$
$$\frac{k_c}{u_c^2} = 0.067$$

in spite of some quantitative discrepancies.

It should be noted that an important problem is the determination of the number of bands n with which the flow region is overlaid in the process of numerical analysis. In the present calculations the value $n = 100$ was selected. A further increase in the number of bands did not lead to a substantial refinement of numerical data. The study of the effect of the size $\Delta\eta = 1/n$ showed that a suitable accuracy may be attained at a smaller n. Actually, calculations with $n = 80$ yield results differing from those with $n = 100$ by less than 1%.

CONCLUSION

In the present paper a new method is suggested for numerical integration of the system of partial differential equations describing the development of a plane vertical turbulent jet in a stagnant non-stratified medium. The method is based on mathematical variables that make it possible to write the starting equations in a form convenient for numerical analysis. As a result, the problem of the search for the unknown functions is reduced to the solution of a system of first-order ordinary differential equations by the Runge–Kutta method. This enables one not only to raise the efficiency but also to avoid the difficulties inherent in numerical calculations by means of ordinary finite-difference schemes. Finally, it should be noted that the proposed method can also be used for numerical investigation of other types of jet flows.

REFERENCES

1. Ya. B. Zeldovich, The limiting zones of freely ascending convective flows, *Zh. Eksp. Teor, Fiz.* **7**(12), 1463–1465 (1937).
2. V. N. Korovkin, A turbulent convective jet over a linear heat source, *Izv. Akad. Nauk. BSSR, Ser. Fiz.-Energ. Nauk* No. 4, 84–88 (1989).
3. O. G. Martynenko, V. N. Korovkin and Yu. A. Sokovishin, Asymptotic investigation of turbulent buoyant jets, *Wärme- und Stoffübertrag.* **21**(4), 199–207 (1987).
4. Ye. V. Bruyatskiy, *Turbulent Stratified Jet Flows*. Izd. Naukova Dumka, Kiev (1986).
5. O. G. Martynenko, V. N. Korovkin and Yu. A. Sokovishin, *The Theory of Buoyancy Jets and Wakes*. Izd. Nauka Teknika, Minsk (1991).
6. J. F. Sini and I. Dekeyser, Numerical prediction of turbulent plane jets and forced plume by use of the k–ε model of turbulence, *Int. J. Heat Mass Transfer* **30**, 1787–1801 (1987).
7. B. R. Ramaprian and M. S. Chandrasekhara, Measurements in vertical plane turbulent plumes, *Trans. ASME, J. Fluid Engng* **111**, 69–77 (1989).
8. B. R. Ramaprian and M. S. Chandrasekhara, LDA measurements in plane turbulent jets, *Trans. ASME, J. Fluid Engng* **107**, 264–271 (1985).
9. L. J. S. Bradbury, The structure of a self-preserving turbulent plane jet, *J. Fluid Mech.* **23**, 31–61 (1965).

ETUDE NUMERIQUE DE JETS TURBULENTS PLANS ET FLOTTANTS

Résumé—Le modèle k–ε de turbulence est utilisé pour calculer les champs dynamique et thermique dans des jets plans verticaux turbulents dans un environnement uniformément au repos. On suggère une nouvelle approche efficace, basée sur l'introduction de variables mathématiques, pour résoudre le système d'équations aux dérivées partielles (continuité, conservation de la quantité de mouvement, de la chaleur, de l'énergie cinétique turbulente et de son taux de dissipation). Une comparaison est faite entre les résultats des présents calculs et les données expérimentales d'autres auteurs, et on constate un accord satisfaisant.

NUMERISCHE UNTERSUCHUNG EINES TURBULENTEN EBENEN AUFTRIEBSSTRAHLS

Zusammenfassung—Mit Hilfe des k–ε Turbulenzmodells wird das Geschwindigkeitsfeld und das Temperaturfeld in einem ebenen turbulenten senkrechten Strahl in einer ruhenden Umgebung berechnet. Das System partieller Differentialgleichungen, das sich aus den Erhaltungssätzen für Masse, Impuls, Wärme, turbulente kinetische Energie sowie die Dissipationsrate ergibt, wird auf der Grundlage mathematischer Variabler ein neues Lösungsverfahren vorgeschlagen. Die Ergebnisse dieser Berechnungen werden mit Versuchswerten anderer Autoren verglichen, wobei sich eine zufriedenstellende Übereinstimmung ergibt.

ЧИСЛЕННОЕ ИССЛЕДОВАНИЕ ТУРБУЛЕНТНЫХ ПЛОСКИХ ВЫНУЖДЕННЫХ И ПЛАВУЧИХ СТРУЙ

Аннотация—Использована k–ε модель турбулентности для расчета скоростных и температурных полей в плоских турбулентных вертикальных струях в однороднй неподвижной среде. Предложен новый эффективный подход к решению определяющей системы дифференциальных уравнений в частных производных (неразрывности, сохранения импульса, тепла, турбулентной кинетической энергии и скорости ее диссипации), основанный на введении математических переменных. Проведено сравнение результатов настоящих расчетов с экспериментальными данными других авторов и обнаружено их удовлетворительное согласие.